

On the Quasi-Isotropic Inflationary Solution

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In this paper we find a solution for a quasi-isotropic inflationary Universe which allows to introduce in the problem a certain degree of inhomogeneity. We consider a model which generalizes the (flat) FRW one by introducing a first order inhomogeneous term, whose dynamics is induced by an effective cosmological constant. The 3-metric tensor is constituted by a dominant term, corresponding to an isotropic-like component, while the amplitude of the first order one is controlled by a “small” function $\eta(t)$.

In a Universe filled with ultra relativistic matter and a real self-interacting scalar field, we discuss the resulting dynamics, up to first order in η , when the scalar field performs a slow roll on a plateau of a symmetry breaking configuration and induces an effective cosmological constant.

We show how the spatial distribution of the ultra relativistic matter and of the scalar field admits an arbitrary form but nevertheless, due to the required inflationary e-folding, it cannot play a serious dynamical role in tracing the process of structures formation (via the Harrison–Zeldovic spectrum). As a consequence, this paper reinforces the idea that the inflationary scenario is incompatible with a classical origin of the large scale structures.

1 General Statement

The inflationary model [1]-[5] is, up to now, the most natural and complete scenario to make account of the problems outstanding in the Standard Cosmological Model, like the horizons and flatness paradoxes [6] (for pioneer works on inflationary scenario and the spectrum of gravitational perturbation, see also [7, 8]); indeed such a dynamical scheme, on one hand is able to justify the high isotropy of the cosmic microwaves background radiation (characterized by temperature fluctuations $\mathcal{O}(10^{-4})$ [9]) and, on the other one, provides a mechanism for generating a (scale invariant) spectrum of inhomogeneous perturbations (via the scalar field quantum fluctuations).

Moreover, as shown in [10, 11], a slow-rolling phase of the scalar field allows to connect the generic inhomogeneous Mixmaster dynamics [12]-[18] with a later quasi-isotropic Universe evolution (in principle compatible with the actual cosmological picture).

With respect to this, we investigate the dynamics performed by small inhomogeneous corrections to a leading order metric, during inflationary expansion.

The model presented has the relevant feature to contain inhomogeneous corrections to a flat FRW Universe, which in principle could take a role to understand the process of structures formation, even in presence of an inflationary behavior; however, a careful analysis of our result prevents this possibility in view of the strong inflationary e-folding, so confirming the expected incompatibility between an inflationary scenario and a classical origin of the Universe clumpiness.

In what follows, we will use the so-called *quasi isotropic solution* which was introduced in [19] as the simplest, but rather general, extension of the FRW model; for a discussion of the quasi isotropic solution in the framework of the “long-wavelength” approximation, see [20] while for the implementation of such solution after inflation to generic equation of state and to the case of two ideal hydrodynamic fluid see, respectively, [21] and [22]. In [23] (see also [24]) this solution is discussed in the presence of a real scalar field kinetic energy, which leads to a power-law solution for the 3-metric, and predicts interesting features for the ultra relativistic matter dynamics.

In the present paper we analyse the opposite dynamical scheme, when the scalar field undergoes a slow-rolling phase since the effective cosmological constant dominates its kinetic energy. We provide, up to the first two orders of approximation and in a synchronous reference, a detailed description of the 3-metric, of the scalar field and of the ultra relativistic matter dynamics, showing that the volume of the Universe expands exponentially and induces a corresponding exponential decay (as the inverse fourth power of the cosmic scale factor), either of the 3-metric corrections, as well as of the ultra relativistic matter (the same behavior characterizes roughly even the scalar field inhomogeneities). It is remarkable that the spatial dependence of such component is described by a function which remains an arbitrary degree of freedom; in spite of such freedom in fixing the primordial spectrum of inhomogeneities, due to the inflationary e-folding, we show there is no chance that, after the de-Sitter phase, such relic perturbations can survive enough to trace the large scale structures formation by an Harrison–Zeldovic spectrum.

This behavior suggests that the spectrum of inhomogeneous perturbations [25] cannot

arise, in such a model, directly by the classical field nature, instead of by its quantum dynamics.

Finally, we recall that the presence of the kinetic term of a scalar field, here regarded as negligible, induces, near enough to the singularity, a deep modification of the general cosmological solution, leading to the appearance of a dynamical regime, during which, point by point in space, the three spatial directions behaves monotonically [26, 27].

In Section 2 is presented a brief review concerning the origin of spectrum perturbations in the inflationary scenario. We review in Section 3 the general formulation of the cosmological problem corresponding to the dynamics of a tridimensional Universe filled of ultra relativistic matter, and in which lives a real self-interacting scalar field. In Section 4 it is introduced the quasi-isotropic formalism, which is applied in Section 5 to get, far enough from the singularity, the inflationary solution; finally, in Section 6 follow some concluding remarks and physical considerations on the model obtained.

2 Inhomogeneous Perturbations from an Inflationary Scenario

The theory of inflation is based on the idea that during the Universe evolution takes place a phase transition (for instance associated with a spontaneous symmetry breaking of a Grand-Unification model of strong and electroweak interactions) which induces an effective cosmological constant dominating the expansion dynamics. As a result arises an exponential expansion of the Universe and, under a suitable fine-tuning of the parameters, it is able to “stretch” so strongly the geometry that two fundamental and puzzling problems of the Standard Cosmological Model, like the ones of the so-called *horizons* and *flatness paradoxes* are naturally solved.

In the “new inflation” theory, the Universe undergoes a de-Sitter phase when the scalar field performs a “slow-rolling” behavior over a very flat region of the potential between the false and true vacuum. Such an exponential expansion ends with the scalar field falling down in the potential well associated to the real vacuum. Here the scalar field dies via damped (by the expansion of the Universe and particles creation) oscillations which reheat the cold Universe left by the de-Sitter expansion (we recall that the relativistic particles temperature is proportional to the inverse scale factor). Indeed, the decay of this super-cooled bosons condensate into relativistic particles, being a typical irreversible process, generates a huge amount of entropy, which allows to account for the present high value ($\sim \mathcal{O}(10^{88})$) of the Universe entropy per comoving volume.

Apart from the transition across the potential barrier between false and true vacuum, which takes place in general via a tunneling, the whole inflationary dynamics can be satisfactorily described via a classical uniform scalar field $\phi = \phi(t)$. The assumption that the field behaves in a classical way is supported by its bosonic and cosmological nature, but the existence of quantum fluctuations of the field within the different inflationary “bubbles” leads to relax the hypothesis of dealing with a perfectly uniform scalar field.

In general, when analyzing density perturbations, turns out convenient to introduce

the dimensionless quantity

$$\delta\rho(t, x^\gamma) \equiv \frac{\Delta\rho(t, x^\gamma)}{\bar{\rho}} = \frac{\rho - \bar{\rho}}{\bar{\rho}}, \quad (1)$$

where $\bar{\rho}$ denotes the mean density and $\gamma = 1, 2, 3$. The best formulation of the density perturbations theory is obtained expanding $\delta\rho$ in its Fourier components, or modes,

$$\delta\rho_k = \frac{1}{(2\pi)^3} \int d^3x e^{ik_\alpha x^\alpha} \delta\rho(t, x^\gamma). \quad (2)$$

As long as the perturbations are in the linear regime, i.e. $\delta\rho_k \ll 1$, we can follow appropriately the dynamics of each mode with wavenumber k , which corresponds to a wavelength $\lambda = \frac{2\pi}{k}$; however, in an expanding Universe, the *physical* size of the perturbations evolves via the *cosmic scale factor* $a(t)$,

Since in the Standard Cosmological Model, the ‘‘Hubble radius’’ scales as $H^{-1} \propto t$, while $a(t) \propto t^n$ with $n < 1$, then every perturbation, now inside the Hubble radius, was outside it at some earlier time. We stress how the perturbations with a physical size, respectively smaller or greater than the Hubble radius, have a very different dynamics, the former ones being affected by the action of the microphysics processes.

In the case of an inflationary scenario, the situation is quite different. Since during the de-Sitter phase the Hubble radius remains constant, while the cosmic scale factor ‘‘explodes’’ exponentially; hence, all cosmological interesting scales have crossed the horizon twice, i.e. the perturbations begin sub-horizon sized, cross the Hubble radius during inflation and later cross back again inside the horizon.

This feature has a strong implication on the initial spectrum of density perturbations predicted by inflation. We present a qualitative argument to understand how this spectrum can be generated.

During inflation, the density perturbations are expected to arise from the quantum mechanical fluctuations of the scalar field ϕ ; these are, as usual, decomposed in their Fourier components $\delta\phi_k$, i.e.

$$\delta\phi_k = \frac{1}{(2\pi)^3} \int d^3x e^{ik_\alpha x^\alpha} \delta\phi(t, x^\gamma), \quad (3)$$

The spectrum of quantum mechanical fluctuations of the scalar field is defined as

$$(\Delta\phi)_k^2 \equiv \frac{1}{\mathcal{V}} \frac{k^3}{2\pi^2} |\delta\phi_k|^2, \quad (4)$$

where \mathcal{V} denotes the comoving volume. For a massless minimally coupled scalar field in a de-Sitter space-time, which approximates very well the real physical situation during the Universe exponential expansion, it is well known that (see [6] §8.4 and references therein)

$$(\Delta\phi)_k^2 = \left(\frac{H}{2\pi}\right)^2, \quad (5)$$

then the mean square fluctuation of ϕ , $(\Delta\phi)^2$, takes the following form:

$$(\Delta\phi)^2 = \frac{1}{(2\pi)^3 V} \int d^3k |\delta\phi_k|^2 = \int \left(\frac{H}{2\pi}\right)^2 d(\ln k). \quad (6)$$

Since H is constant during the de-Sitter phase of the Universe, each mode k contributes roughly the same amplitude to the mean square fluctuation. Indeed, the only dependence on k takes place in the logarithmic term, but the modes of cosmological interest lay between 1 Mpc and 3000 Mpc (it is commonly adopted the convention to set the actual cosmic scale factor equal to unity), corresponding to a logarithmic interval of less than an order of magnitude.

Thus we can conclude that any mode k crosses the horizon having almost a constant amplitude $\delta\phi_k \simeq H/2\pi$. A delicate question concerns the mechanism by which such quantum fluctuations of the scalar field achieve a classical nature (for a detailed discussion see [28]); here we simply observe how each mode k , once reached a classical stage, is subjected to the dynamics

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0; \quad (7)$$

according to this equation, super-horizon modes $k \ll aH$ (i.e. $\lambda_{phys} \gg H^{-1}$) admit the trivial dynamics (7) with $\delta\phi_k \sim \text{const.}$. This simple analysis implies the important feature that any mode re-enters the horizon with roughly the same amplitude it had at the first horizon crossing. The spectrum of perturbations so generated is then induced into the relativistic energy density coming from the reheating phase, associated with the bosons decay; since that moment the evolution of the perturbation spectrum follows a standard paradigm.

Finally, we discuss another feature of the quantum mechanical fluctuations generated by the inflation, regarding their gaussian distribution: as long as the field ϕ is minimally coupled, it has a low self interaction and each mode fluctuates independently; hence, since the fluctuations we actually observe are the sum of many of its quantum ones, their distribution is expectable to be gaussian (as it should be for the sum of many independent variables).

3 Field Equations in the Synchronous Reference

The line element in a synchronous reference frame of coordinates (t, x^γ) (we adopt units in which the speed of light c is equal to unity)¹ writes

$$ds^2 = dt^2 - \gamma_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta, \quad (8)$$

where $\gamma_{\alpha\beta}(t, x^\gamma)$ is the three-dimensional metric tensor describing the geometry of the spatial slices.

¹Greek indexes α, β, γ take values 1,2,3 labeling the spatial coordinates on the space-like hypersurfaces of constant proper cosmological time t , while Latin ones label 0 to 3.

Let us describe the matter by a perfect fluid with ultra relativistic equation of state $p = \frac{\epsilon}{3}$ (p and ϵ denote respectively the fluid pressure and energy density) and a scalar field $\phi(t, x^\gamma)$ with a potential term $V(\phi)$.

In what follows, we write the Einstein equations as

$$R_i^k = \chi \sum_{(z)=m,\phi} \left(T_i^{k(z)} - \frac{1}{2} \delta_i^k T_l^{l(z)} \right) \quad (9)$$

where χ denotes the Einstein constant $\chi = 8\pi G$ (G being the Newton constant) and $T_i^{k(m)}$ and $T_i^{k(\phi)}$ indicate, respectively, the energy-momentum tensor of the matter and the scalar field. Explicitly, in a synchronous reference, such equations reduce to the partial differential system

$$\frac{1}{2} \partial_t k_\alpha^\alpha + \frac{1}{4} k_\alpha^\beta k_\beta^\alpha = \chi \left[-(4u_0^2 - 1) \frac{\epsilon}{3} - (\partial_t \phi)^2 + V(\phi) \right] \quad (10)$$

$$\frac{1}{2} (k_{\alpha;\beta}^\beta - k_{\beta;\alpha}^\beta) = \chi \left(\frac{4}{3} \epsilon u_\alpha u_0 + \partial_\alpha \phi \partial_t \phi \right) \quad (11)$$

$$\frac{1}{2\sqrt{\gamma}} \partial_t (\sqrt{\gamma} k_\alpha^\beta) + P_\alpha^\beta = \chi \left[\gamma^{\beta\gamma} \left(\frac{4}{3} \epsilon u_\alpha u_\gamma + \partial_\alpha \phi \partial_\gamma \phi \right) + \left(\frac{\epsilon}{3} + V(\phi) \right) \delta_\alpha^\beta \right], \quad (12)$$

where the vector field u_i ($i = 0, \dots, 3$) represents the matter four-velocity and we used the notations

$$\partial_t(\quad) \equiv \frac{\partial(\quad)}{\partial t}, \quad \partial_\alpha(\quad) \equiv \frac{\partial(\quad)}{\partial x^\alpha}, \quad (13)$$

$$\gamma \equiv \det \gamma_{\alpha\beta}, \quad k_{\alpha\beta} \equiv \partial_t \gamma_{\alpha\beta}, \quad k_\alpha^\beta = \gamma^{\beta\gamma} k_{\alpha\gamma}. \quad (14)$$

The metric $\gamma_{\alpha\beta}$ allows to construct the three-dimensional Ricci tensor $P_\alpha^\beta = \gamma^{\beta\gamma} P_{\alpha\gamma}$ as

$$P_{\alpha\beta} = \partial_\gamma \lambda_{\alpha\beta}^\gamma - \partial_\alpha \lambda_{\beta\gamma}^\gamma + \lambda_{\alpha\beta}^\gamma \lambda_{\gamma\delta}^\delta - \lambda_{\alpha\delta}^\gamma \lambda_{\beta\gamma}^\delta \quad (15)$$

in which appear the pure spatial Christoffel symbols

$$\lambda_{\alpha\beta}^\gamma \equiv \frac{1}{2} \gamma^{\gamma\delta} (\partial_\alpha \gamma_{\delta\beta} + \partial_\beta \gamma_{\alpha\delta} - \partial_\delta \gamma_{\alpha\beta}) \quad (16)$$

also used to form the covariant derivative $(\quad)_{;\alpha}$.

The dynamics of the scalar field $\phi(t, x^\gamma)$ is described by a partial differential equation, coupled to the above Einsteinian system, which in a synchronous reference reads

$$\partial_{tt} \phi + \frac{1}{2} k_\alpha^\alpha \partial_t \phi - \gamma^{\alpha\beta} \phi_{;\alpha;\beta} + \frac{dV}{d\phi} = 0 \quad (17)$$

where we adopted the obvious notation

$$\partial_{tt}(\quad) \equiv \frac{\partial^2(\quad)}{\partial t^2}. \quad (18)$$

The hydrodynamic equations, taking into account for the matter evolution, in a synchronous reference and for the ultra relativistic case, possess the structure [19]

$$\frac{1}{\sqrt{\gamma}}\partial_t(\sqrt{\gamma}\epsilon^{3/4}u_0) + \frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\epsilon^{3/4}u^\alpha) = 0 \quad (19)$$

$$4\epsilon\left(\frac{1}{2}\partial_t u_0^2 + u^\alpha\partial_\alpha u_0 + \frac{1}{2}k_{\alpha\beta}u^\alpha u^\beta\right) = (1 - u_0^2)\partial_t\epsilon - u_0 u^\alpha\partial_\alpha\epsilon \quad (20)$$

$$4\epsilon\left(u_0\partial_t u_\alpha + u^\beta\partial_\beta u_\alpha + \frac{1}{2}u^\beta u^\gamma\partial_\alpha\gamma_{\beta\gamma}\right) = -u_\alpha u_0\partial_t\epsilon + (\delta_\alpha^\beta - u_\alpha u^\beta)\partial_\beta\epsilon. \quad (21)$$

In view of the chosen feature for (9), equation (19) doesn't contain spatial gradients of the 3-metric tensor and of the scalar field. This scheme is completed by observing how it be covariant with respect to coordinates transformation of the form

$$t' = t + f(x^\gamma), \quad x^{\alpha'} = x^{\alpha'}(x^\gamma) \quad (22)$$

being f a generic space dependent function.

4 Quasi isotropic Model

In order to introduce in a quasi isotropic (inflationary) scenario small inhomogeneous corrections to the leading order, we require a tridimensional metric tensor having the following structure

$$\begin{aligned} \gamma_{\alpha\beta}(t, x^\gamma) &= a^2(t)\xi_{\alpha\beta}(x^\gamma) + b^2(t)\theta_{\alpha\beta}(x^\gamma) + \mathcal{O}(b^2) = \\ &= a^2(t)\left[\xi_{\alpha\beta}(x^\gamma) + \eta(t)\theta_{\alpha\beta}(x^\gamma) + \mathcal{O}(\eta^2)\right] \end{aligned} \quad (23)$$

where we defined $\eta \equiv \frac{b^2}{a^2}$ and suppose that η satisfies the condition

$$\lim_{t \rightarrow \infty} \eta(t) = 0. \quad (24)$$

We shall analyse the field equations (10)-(12) retaining only terms linear in η and its time derivatives. In the limit of the considered approximation, the inverse three-metric reads

$$\gamma^{\alpha\beta}(t, x^\gamma) = \frac{1}{a^2(t)}\left(\xi^{\alpha\beta}(x^\gamma) - \eta(t)\theta^{\alpha\beta}(x^\gamma) + \mathcal{O}(\eta^2)\right), \quad (25)$$

where $\xi^{\alpha\beta}$ denotes the inverse matrix of $\xi_{\alpha\beta}$ and assumes a metric role, i.e. we have

$$\xi^{\beta\gamma}\xi_{\alpha\gamma} = \delta_\alpha^\beta \quad \theta^{\alpha\beta} = \xi^{\alpha\gamma}\xi^{\beta\delta}\theta_{\gamma\delta}. \quad (26)$$

The covariant and contravariant three-metric expressions lead to the important explicit relations

$$k_\alpha^\beta = 2\frac{\dot{a}}{a}\delta_\alpha^\beta + \dot{\eta}\theta_\alpha^\beta \quad \Rightarrow \quad k_\alpha^\alpha = 6\frac{\dot{a}}{a} + \dot{\eta}\theta \quad \theta \equiv \theta_\alpha^\alpha \quad (27)$$

where we set $(\quad)^\cdot \equiv d(\quad)/dt$.

Since it should take place the fundamental equality $\partial_t(\ln \gamma) = k_\alpha^\alpha$, then we immediately get

$$\gamma = ja^6 e^{\eta\theta} \Rightarrow \sqrt{\gamma} = \sqrt{ja^3} e^{\frac{1}{2}\eta\theta} \sim \sqrt{ja^3} \left(1 + \frac{1}{2}\eta\theta + \mathcal{O}(\eta^2)\right), \quad j \equiv \det \xi_{\alpha\beta}. \quad (28)$$

Equations (10)-(12) are analysed via the standard procedure of constructing asymptotic solutions in the limit $t \rightarrow \infty$, by verifying *a posteriori* the self-consistency of the approximation scheme, i.e. that the neglected terms were really of higher order in time.

5 Inflationary Solution

In the quasi-isotropic approach, under consideration, we assume that the scalar field dynamics, in the plateau region, be governed by a potential term as

$$V(\phi) = \Lambda + K(\phi), \quad \Lambda = \text{const.} \quad (29)$$

where Λ is the dominant term and $K(\phi)$ is a small correction to it. The role of K , as shown in the following, is to contain inhomogeneous corrections via the ϕ -dependence; the functional form of K can be any one of the most common inflationary potentials, as they appear near the flat region for the evolution of ϕ .

What follows remains valid, for example, for the relevant cases of the quartic and Coleman–Weinberg expressions

$$K(\phi) = \begin{cases} -\frac{\lambda}{4}\phi^4, & \lambda = \text{const.} \\ B\phi^4 \left[\ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2} \right], & \sigma = \text{const.}, \end{cases} \quad (30)$$

viewed as corrections to the constant Λ term, though explicit calculation are below developed only for the first case.

Our inflationary solution is obtained under the standard requirements

$$\frac{1}{2}(\partial_t \phi)^2 \ll V(\phi) \quad (31)$$

$$|\partial_{tt} \phi| \ll |k_\alpha^\alpha \partial_t \phi|. \quad (32)$$

The above approximations and the substitution of (27) reduce the scalar field equation (17) to the form

$$\left(3\frac{\dot{a}}{a} + \frac{1}{2}\dot{\eta}\theta\right) \partial_t \phi - \lambda \phi^3 = 0 \quad (33)$$

where we assumed that the contribution of the ϕ spatial gradients be negligible.

Similarly, the quasi-isotropic approach (in which the inhomogeneities become relevant only for the next approximation order), once neglecting the spatial derivatives, in (19), leads to

$$\sqrt{\gamma} \epsilon^{3/4} u_0 = l(x^\gamma) \Rightarrow \epsilon \sim \frac{l^{4/3}}{j^{2/3} a^4 u_0^{4/3}} \left(1 - \frac{2}{3}\eta\theta + \mathcal{O}(\eta^2)\right) \quad (34)$$

where $l(x^\gamma)$ denotes an arbitrary function of the spatial coordinates.

Let us now face, in the same approximation scheme, the analysis of the Einstein equations (10)-(12). Taking into account (31), up to the first order in η , equation (10) reads

$$3\frac{\ddot{a}}{a} + \left[\frac{1}{2}\ddot{\eta} + \frac{\dot{a}}{a}\dot{\eta}\right]\theta - \chi\Lambda = -\chi\frac{\epsilon}{3}(3 + 4u^2) \quad (35)$$

having set

$$u^2 \equiv \frac{1}{a^2}\xi^{\alpha\beta}u_\alpha u_\beta \quad \Rightarrow \quad u_0 = \sqrt{1 + u^2}. \quad (36)$$

Equation (12) reduces to the form

$$\begin{aligned} & \frac{2}{3}(a^3)^{\cdot\cdot}\delta_\alpha^\beta + (a^3\dot{\eta})^{\cdot}\theta_\alpha^\beta + \frac{1}{3}[(a^3)^{\cdot}\eta]^{\cdot}\theta\delta_\alpha^\beta + aA_\alpha^\beta = \\ & = \chi \left[\frac{1}{a^2}(\xi^{\beta\gamma} - \eta\theta^{\beta\gamma}) \frac{4}{3}\epsilon u_\alpha u_\gamma + \left(\frac{\epsilon}{3} + \Lambda\right)\delta_\alpha^\beta \right] 2a^3 \left(1 + \frac{\eta\theta}{2}\right). \end{aligned} \quad (37)$$

In this expression, the spatial curvature term reads, in the leading order, as

$$P_\alpha^\beta(t, x^\gamma) = \frac{1}{a^2(t)}A_\alpha^\beta(x^\gamma), \quad (38)$$

where $A_{\alpha\beta}(x^\gamma) = \xi_{\beta\gamma}A_\alpha^\gamma$ denotes the Ricci tensor corresponding to $\xi_{\alpha\beta}(x^\gamma)$.

The trace of (37) gives the additional relation

$$2(a^3)^{\cdot\cdot} + (a^3\eta)^{\cdot\cdot}\theta + aA_\alpha^\alpha = \chi \left[\frac{\epsilon}{3}(3 + 4u^2) + 3\Lambda \right] 2a^3 \left(1 + \frac{\eta\theta}{2}\right). \quad (39)$$

Comparing (35) with the trace (39), via their common term $(3 + 4u^2)\epsilon/3$, and estimating the different orders of magnitude, we get the following equations

$$(a^3)^{\cdot\cdot} + 3a^2\ddot{a} - 4\chi a^3\Lambda = 0 \quad (40)$$

$$A_{\alpha\beta} = 0 \quad (41)$$

$$3(a^3\eta)^{\cdot\cdot} + 3a^3\ddot{\eta} + 2(a^3)^{\cdot}\dot{\eta} + 9a^2\ddot{a}\eta - 12\chi a^3\Lambda\eta = 0. \quad (42)$$

Since (41) implies that the the tridimensional Ricci tensor vanishes, and this condition corresponds to the vanishing of the Riemann tensor too, then we can conclude that the obtained Universe is flat up to the leading order, i.e.

$$\xi_{\alpha\beta} = \delta_{\alpha\beta} \quad \Rightarrow \quad j = 1. \quad (43)$$

Equation (40) admits the expanding solution

$$a(t) = a_0 \exp \left\{ \frac{\sqrt{3\chi\Lambda}}{3} t \right\} \quad (44)$$

being a_0 , the initial value of the scale factor amplitude, taken at the instant $t = 0$ when the de-Sitter phase starts.

Expression (44) for $a(t)$, when substituted in (42) yields the differential equation for η

$$\ddot{\eta} + \frac{4}{3}\sqrt{3\chi\Lambda} \dot{\eta} = 0, \quad (45)$$

whose only solution, satisfying the limit (24), reads

$$\eta(t) = \eta_0 \exp \left\{ -\frac{4}{3}\sqrt{3\chi\Lambda} t \right\} \quad \Rightarrow \quad \eta = \eta_0 \left(\frac{a_0}{a} \right)^4, \quad (46)$$

and, of course, we require $\eta_0 \ll a_0$.

Equations (34) and (35), in view of the solutions (44) for $a(t)$ and (46) for $\eta(t)$, are matched with consistency, by posing

$$\begin{aligned} u_\alpha(t, x^\gamma) &= v_\alpha(x^\gamma) + \mathcal{O}(\eta^2) \\ (u_0)^2 &= 1 + \mathcal{O}\left(\frac{1}{a^2}\right) \approx 1, \end{aligned} \quad (47)$$

and respectively

$$\epsilon = -\frac{4}{3}\Lambda\eta\theta, \quad (48)$$

which implies $\theta < 0$ for each point of the allowed domain of the spatial coordinates. The comparison between (34) and (48) leads to the explicit expression also for $l(x^\gamma)$ in terms of θ

$$l(x^\gamma) = \left(\frac{4}{3}\Lambda\eta_0 a_0^4 \right)^{3/4} (-\theta)^{3/4}. \quad (49)$$

Defining the auxiliary tensor with unit trace $\Theta_{\alpha\beta}(x^\gamma) \equiv \theta_{\alpha\beta}/\theta$, the above analysis permits, from (37), to obtain for it the expression

$$\Theta_\alpha^\beta = \frac{\delta_\alpha^\beta}{3}. \quad (50)$$

By (33), the explicit form for a , once expanded in η , yields the first two leading orders of approximation for the scalar field

$$\phi(t, x^\gamma) = \mathcal{C} \sqrt{\frac{t_r}{t_r - t}} \left(1 - \frac{1}{4\sqrt{3\chi\Lambda}} \frac{\eta}{t_r - t} \theta \right), \quad t_r = \frac{\sqrt{3\chi\Lambda}}{\mathcal{C}^2 2\lambda}, \quad (51)$$

where \mathcal{C} is an integration constant; finally, equation (12) provides v_α in terms of θ

$$v_\alpha = -\frac{3}{4} \frac{1}{\sqrt{3\chi\Lambda}} \partial_\alpha \ln |\theta|. \quad (52)$$

On the basis of (50)-(52), the hydrodynamic equations (19)-(21) reduce to an identity, in the leading order of approximation; in fact such equations contain the energy density of the ultra relativistic matter, which is known only in the first order (the higher one of the Einstein equations). Therefore it makes no sense to take into account higher order contributions, coming from those equations.

As soon as $(t_r - t)$ is sufficiently large, it can be easily checked that the solution here constructed is completely self-consistent to the all calculated orders of approximation in time and contains one physically arbitrary function of the spatial coordinates, $\theta(x^\gamma)$ which, indeed, being a three-scalar, is not affected by spatial coordinate transformations. In particular, the terms quadratic in the spatial gradients of the scalar field are of order

$$(\partial_\alpha \phi)^2 \approx \mathcal{O} \left(\frac{\eta^2}{a^2} \frac{1}{(t_r - t)^3} \right) \quad (53)$$

and therefore can be neglected with respect to all the inhomogeneous ones.

Such solution fails when t approaches t_r and therefore its validity requires that the de-Sitter phase ends (with the fall of the scalar field in the true potential vacuum) when t is yet much smaller than t_r (see below).

6 Physical considerations

The peculiar feature of the solution above constructed lies in the independence of the function θ which, from a cosmological point of view, implies the existence of a quasi-isotropic inflationary solution in correspondence to an arbitrary spatial distribution of ultra relativistic matter and of the scalar field.

We get an inflationary picture from which the Universe outcomes with the appropriate standard features, but in presence of a suitable spectrum of *classical* perturbations as due to the small inhomogeneities which are modelizable according to an Harrison–Zeldovic spectrum; in fact, expanding the function θ in Fourier series as

$$\theta(x^\gamma) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \tilde{\theta}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d^3k, \quad (54)$$

we can impose an Harrison–Zeldovic spectrum by requiring

$$|\tilde{\theta}|^2 = \frac{Z}{|k|^{3/2}}, \quad Z = \text{const.} \quad (55)$$

However, the following three points have to be taken into account to give a complete picture for our analysis:

- (i) limiting (as usual) our attention to the leading order, the validity of the slow-rolling regime is ensured by the natural conditions

$$\mathcal{O} \left(\sqrt{\chi \Lambda} (t - t_r) \right) \ll 1, \quad \lambda \gg \mathcal{O}(\chi^2 \Lambda), \quad (56)$$

which respectively translate (32) and (31);

- (ii) denoting by t_i and t_f respectively the beginning and the end of the de-Sitter phase, we should have $t_r \gg t_f$ and the validity of our solution is guaranteed if

(a) the flatness of the potential is preserved, i.e. $\lambda\phi^4 \ll \Lambda$: such a requirement coincides, as it should, with the second of inequalities (56);

(b) given Δ as the width of the flat region of the potential, we require that the de-Sitter phase ends before t becomes comparable with t_r , i.e.

$$\phi(t_f) - \phi(t_i) \sim \sqrt{\frac{\sqrt{\chi\Lambda}}{\lambda}} \frac{t_f - t_i}{t_r^{3/2}} \sim \mathcal{O}(\Delta), \quad (57)$$

where we expanded the solution at the first order in $t_{i,f}/t_r$; via the usual position $(t_f - t_i) \sim \mathcal{O}(10^2)/\sqrt{\chi\Lambda}$, the relation (57) becomes a constraint for the integration constant t_r .

- (iii) In order to get an inflationary scenario, able to overcome the shortcomings present in the Standard Cosmological Model, we need an exponential expansion sufficiently strong. For instance we have to require that a region of space, corresponding to a cosmological horizon $\mathcal{O}(10^{-24}cm)$ when the de-Sitter phase starts, now covers all the actual Hubble horizon $\mathcal{O}(10^{26}cm)$; the redshift at the end of the de-Sitter phase is $z \sim \mathcal{O}(10^{24})$, then we should require $a_f/a_i \sim e^{60} \sim \mathcal{O}(10^{26})$. Let's estimate the densities perturbations (inhomogeneities) at the (matter-radiation) decoupling age ($z \sim \mathcal{O}(10^4)$) as $\delta_{in} \sim \mathcal{O}(10^{-4})$; if we start by this same value at the beginning of inflation (δ_{in}^i), we arrive at the end with $\delta_{in}^f \sim (\eta_f/\eta_i)\delta_{in}^i \sim \mathcal{O}(10^{-100})$. Though these inhomogeneities increase as z^2 once they are at scale greater than the horizon, nevertheless they reach only $\mathcal{O}(10^{-60})$ at the decoupling age. This result provides support to the idea that the spectrum of inhomogeneous perturbations cannot have a classical origin in presence of an inflationary scenario.

In the considerations above developed, we regard the ratio of the inhomogeneous terms ϵ_f and ϵ_i as the quantity $\delta\rho$ defined in Section 2 and now we show how this assumption is (roughly) correct: after the reheating the Universe is dominated by a homogeneous (apart from the quantum fluctuations) relativistic energy density ρ_r to which is superimposed the relic ϵ_f after inflation; therefore we have

$$\delta\rho = \frac{\epsilon_f}{\rho_r} = \frac{\epsilon_f}{\epsilon_i} \frac{\epsilon_i}{\rho_r} = \left(\frac{a_i}{a_f}\right)^4 \frac{\epsilon_i}{\rho_r}. \quad (58)$$

Hence our statement follows as soon as we observe that the inhomogeneous relativistic energy density before the inflation ϵ_i and the uniform one ρ_r , generated by the reheating process, differ by only some orders of magnitude.

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